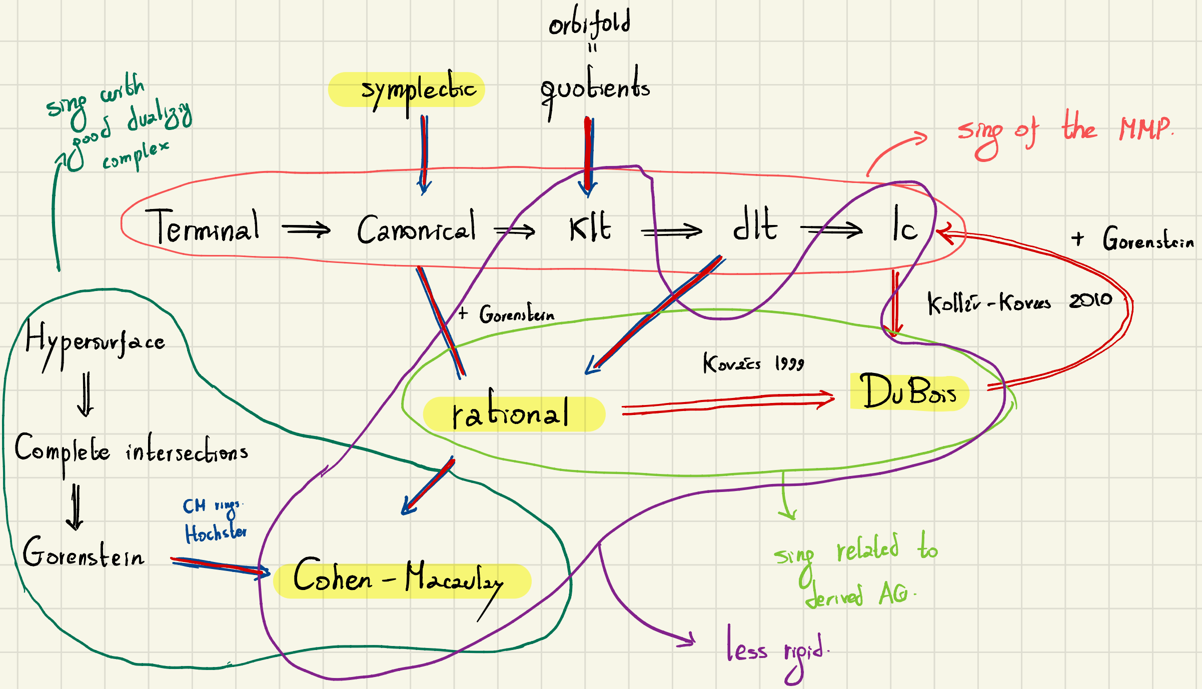
Minimal Model Program Learning Seminar.

Week 7:

- rational singularities.
- log terminal singularities.
- terminal 3-fold singularities.

Geography of singularities:



● : From definition or easy implication.

● : Requires a (short) argument.

● : Requires some real work.

Properties: Closed under quotients? $X \longrightarrow X/G$

Closed under deformations? $H \subseteq X$

klt ↶ klt

when H is general.

Cohen-Macaulay:

(R, m) Noetherian local ring, N be a finite R-module.

$\dim N = \dim \operatorname{supp} N$. N is called Cohen-Macaulay if one of the equivalent conditions holds: (CM for short).

i) There exists $x_1, \dots, x_r \in m$, $r = \dim N$, x_i is not a zero divisor in $N/(x_1, \dots, x_{i-1})N$ for all i. ($x_1, \dots, x_r$ is called a N-regular sequence).

ii) If $x_1, \dots, x_r \in m$ $(r = \dim N)$ and $\dim N/(x_1, \dots, x_r)N = 0$, then $x_1, \dots, x_r$ is an N-reg sequence

A coherent sheaf $\mathcal{F}$ on a scheme X is CM if $\mathcal{F}_x$ is CM over $\mathcal{O}_{X,x}$ for every $x \in X$.

A scheme X is called CM if its structure sheaf $\mathcal{O}_X$ is CM

Serre condition: A sheaf $\mathcal{F}$ on X is said to satisfy S_d if for every $x \in X$, $\mathcal{F}_x$ has a regular sequence of length $\min\{d, \dim \mathcal{O}_{X,x}\}$.

$d = \dim X$, then X is CM $\iff$ $\mathcal{O}_X$ is S_d.

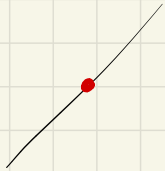
Examples: Normal $\iff$ $R_1 + S_2$, so normal surfaces are CM

If R is CM and G acts on R, then R^G is CM (Hochster - Roberts)

$k[x]/(x^2)$ 0-dim CM.

$k[t^2, t^3]$ 1-dim CM.

$X = \mathrm{Spec}(\mathbb{k}[x,y]/(x^2, xy))$



This is not CM at the origin

Rational singularities: Y is a variety over a field of char 0.

$X \xrightarrow{f} Y$ is a resolution of sing. We say that f is rational if

(1) $f_* \mathcal{O}_X = \mathcal{O}_Y$ (Y normal)

(2) $R^i f_* \mathcal{O}_X = 0$ for $i > 0$. ($i=1$, 1-rational).

We say that Y has rational sing if every resolution is rational

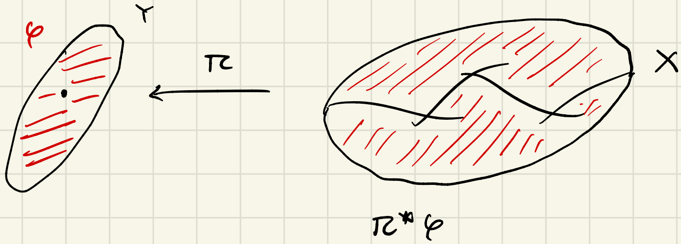
Equivalent: $\mathcal{O}_Y \longrightarrow R^i f_* \mathcal{O}_X$ is a quasi-isomorphism in the der cat.

Examples: A_n singularities are rational $X \xrightarrow{f} Y = A_n$

$f^*(K_Y) = K_X$

Cone over elliptic curve not rational.

Symplectic: $Y \ni y$ symplectic if Y is normal at y and Y^{reg} there a symplectic 2-form which extends on any resolution



DuBois: $X \subseteq Y$ embedding of a scheme into a regular scheme

$Z \longrightarrow Y$ log resolution of X which is an isom outside X.

E the reduced preimage of X in Z. X has DuBois sing

if $\mathcal{O}_X \longrightarrow R\pi_* \mathcal{O}_E$ is a q.i.

Rmk: DB singularities appear often in Hodge theory.

Remarks:
- Serre duality holds for CM sheaves.
- $H \subseteq X$ Cartier & H is CM $\Rightarrow$ X is CM.

Proposition: Y be a variety over a field of char 0.
$f: X \longrightarrow Y$ a resolution. TFAE:

i) f is rational.

ii) Y is CM & $f_* \omega_X = \omega_Y$.

Proof: Y projective, D ample Cartier on Y.

$$H^i(X, \omega_X(rf^*D)) = 0 \quad , i > 0, r > 0.$$

$$H^{n-i}(X, \mathcal{O}_X(-rf^*D)) = 0 \quad i > 0, \quad r > 0.$$

The Leray spectral sequence:

$$E_2^{i,j} = H^i(Y, R^j f_* \mathcal{O}_X(-rD)) \Longrightarrow H^{i+j}(X, \mathcal{O}_X(-rf^*D))$$

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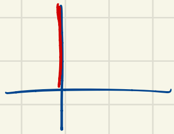
i) $\Longrightarrow$ ii)



L special

By rationality assumption:

$$H^i(Y, \mathcal{O}_Y(-rD)) \simeq H^i(X, \mathcal{O}_X(-rf^*D)) \quad \text{for } i \geq 0$$

Claim: $\underline{H^i(Y, \mathcal{O}_Y(-rD))} = 0$ for $i < n$ & $r \gg 0$

$\Longrightarrow$ Y is CM.

$\Longleftarrow$ is also true (proved later on the book).

Proof: $H \subseteq |r'D|$ general element, H Cartier

$$0 \longrightarrow \mathcal{O}_Y(-(r+r')D) \xrightarrow{\cdot H} \mathcal{O}_Y(-rD) \longrightarrow \mathcal{O}_H(-rD) \longrightarrow 0$$

By the vanishing, we get $H^i(H, \mathcal{O}_H(-rD)) = 0$ $i < n-1$ & $r \gg 0$. Thus, by induction H is CM.

$H \subseteq Y$ is CM & Cartier, hence Y is CM $\square$

$$h^0(Y, \underline{\underline{\omega_Y}}(rD)) = h^0(X, \omega_X(rf^*D)) = h^0(Y, \underline{\underline{f_*\omega_X}}(rD)) \quad r \geq 0$$

This implies $f_*\omega_X = \omega_Y$.

ii) $\Longrightarrow$ i) By induction on the dimension

Claim: $R^i f_* \mathcal{O}_X$ are supported in 0-dim sets.

Proof: $H \subseteq Y$ general, $H' = f^{-1}H$, $f : H' \longrightarrow H$ resolution.

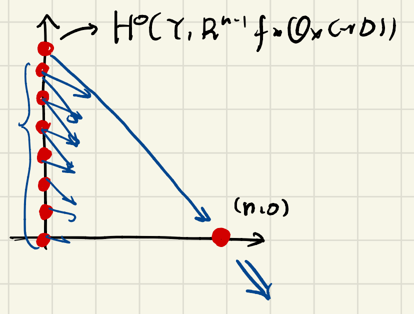
$$f_* \omega_{H'} = f_* (\omega_X(H') \otimes \mathcal{O}_{H'}) = \mathcal{O}_H(H) \otimes f_* \omega_X = \mathcal{O}_H(H) \otimes \overset{\omega_H}{\overset{\parallel}{\omega_Y}}$$

By induction, we see that $\mathcal{O}_H \otimes R^i f_* \mathcal{O}_X = R^i f_* \mathcal{O}_{H'} = 0$

is trivial outside a zero dimensional set.

Therefore $H^p(Y, R^q f_* \mathcal{O}_X(-rD)) = 0$ $p,q>0$ or if $p<n$ & $q=0$.

By the spectral sequence we get:



(*)

$$H^0(Y, R^q f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-rD)) = 0 \quad q<n-1$$

$$H^0(Y, R^{n-1} f_* \mathcal{O}_X(-rD)) \cong$$

$$\ker\left[H^n(Y, \mathcal{O}_Y(-rD)) \xrightarrow{\alpha} H^n(X, \mathcal{O}_X(-rf^*D))\right]$$

$R^q f_* \mathcal{O}_X$ has 0-dim supp & (*) implies that $\overset{q<n-1}{R^q f_* \mathcal{O}_X = 0}$

On the other hand α is the dual to:

$$H^0(Y, \omega_Y(rD)) \longrightarrow H^0(X, \omega_X(rf^*D)) = H^0(Y, f_* \omega_X(rD))$$

since $\omega_Y \simeq f_* \omega_X$. Then α is an isom. $R^{n-1} f_* \mathcal{O}_X = 0$. $\square$

Lemma: (X,Δ) is klt, H is bpf, $H_g \in |H|$ general element.

Then (H, Δ_H) is klt. (same for lc).

Lemma: $Y \xrightarrow{f} X$ finite, $K_Y + \Delta_Y = f^*(K_X+\Delta)$ then

(provided both of them are log pairs).

$$(X,\Delta) \text{ klt} \iff (Y,\Delta_Y) \text{ klt}$$

$$(X,\Delta) \text{ lc} \iff (Y,\Delta_Y) \text{ lc}$$

Idea of the proof: Use Riemann-Hurwitz formula on a log resolution to compare the discrepancies & observe

$$a_E(X,\Delta) = r\, a_E(Y,\Delta_Y)$$

for some positive integer number r. (r is some ram index).

Theorem (Elkik 81): (X,Δ) dlt, then X has rat sing.

Proof: $K_Y \equiv f^*(K_X+\Delta) + A - B$ $\quad Y \longrightarrow X$ log resolution

$\mathrm{Supp}(B) \subseteq \mathrm{Ex}(f)$, $\quad \lfloor A \rfloor = 0$.

By KV vanishing $\underline{R^j f_* \mathcal{O}_Y(\lceil B \rceil) = 0}$ for $j>0$.

$\mathcal{L}$ ample Cartier on X. We have a comm diagram:

$$\begin{tikzcd}
0 = H^i(\mathcal{O}_Y(-rf^*\mathcal{L})) \arrow[r] & H^i(\mathcal{O}_Y(\lceil B \rceil - rf^*\mathcal{L})) \\
H^i(\mathcal{O}_X(-r\mathcal{L})) \arrow[u] \arrow[r, equal] & H^i(\mathcal{O}_X(-r\mathcal{L})) \arrow[u, "\beta"']
\end{tikzcd}$$

β is an isomorphism by spectral sequence.

$$H^i(X, \mathcal{O}_X(-r\mathcal{L}) \otimes R^j f_* \mathcal{O}_X(\lceil B \rceil)) \Longrightarrow H^{i+j}(Y, \mathcal{O}_Y(\lceil B \rceil - rf^*\mathcal{L}))$$

$H^i(\mathcal{O}_Y(-rf^*\mathcal{L})) = 0$ for $\boxed{i<n}$ & $r>0$ by KV vanishing.

We want to conclude that $H^i(\mathcal{O}_X(-r\mathcal{L})) = 0$ for $r\geq 0$, $\boxed{i<n}$

Hence X is CM.

We get the injection

$$H^n(\mathcal{O}_X(-r\mathcal{L})) \hookrightarrow H^n(\mathcal{O}_Y(-rf^*\mathcal{L})).$$

By Serre duality $\quad H^0(Y, \omega_Y(rf^*\mathcal{L})) \twoheadrightarrow H^0(X, \omega_X \otimes \mathcal{O}_X(r\mathcal{L}))$

$$H^0(Y, \omega_Y(rf^*\mathcal{L})) = H^0(X, f_*\omega_Y \otimes \mathcal{O}_X(r\mathcal{L})) \twoheadrightarrow H^0(X, \omega_X \otimes \mathcal{O}_X(r\mathcal{L})) \qquad r \gg 0$$

The surjectivity

$$H^0(X, f_*\omega_Y \otimes \mathcal{O}_X(r\mathcal{L})) \twoheadrightarrow H^0(X, \omega_X \otimes \mathcal{O}_X(r\mathcal{L}))$$

for $r \gg 0$ implies that $f_*\omega_Y \twoheadrightarrow \omega_X$.

So they are isomorphic (rank 1 reflexive sheaves).

Thus, X has rational sing. $\square$

Proposition: X Gorenstein (K_X is Cartier). Rational $\Longleftrightarrow$ Canonical.

Proof: Canonical $\Longrightarrow$ dlt $\Longrightarrow$ rational.

X rational & Gorenstein. $Y \xrightarrow{\pi} X$ a resolution

$$\pi^*(K_X) = K_Y + \underbrace{E - F}$$ integral ≥ 0 with no common support.

If we push-forward $K_Y + E - F$ & $E \neq 0$, then we get associated primes on the image of E. This contradicts $\pi_* \omega_Y \cong \omega_X$. $\square$

Proposition: Symplectic $\Longrightarrow$ rational & Gorenstein.

$$\begin{array}{c} Y \quad \text{resoln} \\ \downarrow \pi \\ X \end{array}$$

Proof: If φ is a 2-form, then φ^r generates the line bundle $\omega_{X_{reg}}$. The fact that $\pi^*\varphi$ extends as a regular holomorphic form $\Longrightarrow$ $\pi^*\varphi^r$ extends.

$\pi^*\omega_X = \omega_Y \otimes \mathcal{O}_Y(-F)$.

Hence, $\pi_*\omega_Y = \pi_*(\pi^*\varphi^r) = \varphi^r = \omega_X$. $\square$

Characterization of Gorenstein:

(R, m) local ring is Gorenstein $\Longleftrightarrow$ there exists a regular sequence $x_1, \dots, x_r$ such that $R/(a_1, \dots, a_r)R$ is Gorenstein 0-dim

Nakayama's Lemma $\Longrightarrow$ $R/(a_1, \dots, a_i)R$ is Gorenstein for every i.

Lemma: Let $(o \in X)$ be an index 1 canonical 3-fold sing. and $o \in H \subseteq X$ a general hyperplane section. Then, either $(o \in H)$ is a Du Val sing or an elliptic sing.

Idea: $H' \xrightarrow{\pi} H$

$\pi_* \omega_{H'} \simeq m\, \omega_H$ $\quad$ $m=1$, then it is DuVal

$m>1$, is an elliptic sing

Theorem: All terminal 3-fold sing of index 1 are cDV. (= one parameter deformation of a Du Val sing).

Symplectic examples:

$\mathbb{C}^n$, $G \leq GL_n(\mathbb{C})$ a finite group.

$\mathbb{C}^n/G$ is a quotient sg. Hence klt.

$G \leq SL_n(\mathbb{C})$, we can show that $\mathbb{C}^n/G$ is a sympl sing

If $x \in X$ is a cone and is symplectic, then
is isomorphic to a Lie group quotient by the smallest
non-zero nilpotent orbit.

Terminal 3-fold singularities:

Theorem: Let $(o \in X)$ be a normal isolated 3-fold sing.

Assume K_X is $\mathbb{Q}$-Cartier of index r and $(\tilde{o} \in \tilde{X}) \xrightarrow{\pi} (o \in X)$ be the index one cover. The group μ_r of r^{th}-roots of unity acts on $\tilde{X}$.

(1) $(o \in X)$ is terminal if and only if a general member $H \in |-K_X|$ containing o is Du Val.

(2) The following is a complete list of all $\tilde{H} := \pi^*(H)$, H and the action of μ_r on $\mathbb{C}^4$.

name	type of $\tilde{H} \longrightarrow H$	r	Type of action
cA/r	$A_{k-1} \longrightarrow A_{kr-1}$	r	$1/r\,(a,-a,1,0;0)$
$cAx/2$	$A_{2k-1} \longrightarrow D_{k+2}$	2	$1/2\,(0,1,1,1;0)$
$cAx/4$	$A_{2k-2} \longrightarrow D_{2k+1}$	4	$1/4\,(1,1,3,2;2)$
$cD/4$	$D_{k+1} \longrightarrow D_{2k}$	2	$1/2\,(1,0,1,1;0)$
$cD/2$	$D_4 \longrightarrow E_6$	3	$1/3\,(0,2,1,1;0)$
$cE/2$	$E_6 \longrightarrow E_7$	2	$1/2\,(1,0,1,1;0)$.

$1/r\,(a_1,\dots,a_4;b)$ means that the erator ξ of μ_r acts on the coordinates $x_1,\dots,x_4$ & on the equation f as:

$$(x_1,\dots,x_4;f) \longmapsto (\xi^{a_1}x_1,\dots,\xi^{a_4}x_4;\xi^b f).$$

Some useful statements about terminal 3-fold sing:

Theorem (Hayakawa): For a terminal 3-fold singularity $P \in X$ of index $r > 1$, there exists a partial resolution

$$X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X \ni P$$

such that X_n is Gorenstein and each $f_i : X_{i+1} \longrightarrow X_i$ is a divisorial contraction to a point of index $r_i > 1$, with extracted divisor of log discrepancy $1/r_i$. All the f_i are weighted blow ups.

Theorem (Kollár–Mori): Let X be a terminal 3-fold and $\varphi : X \longrightarrow Z$ be a flipping contraction. Then, there is a singular point on the flipping locus.

Remark: The above theorem is crucial to prove the termination of terminal 3-fold flips: Terminal 3-folds have isolated singularities. To prove that flips terminate, we will associate a weight function which counts certain contribution from each singular point (this function, called the difficulty function is non-negative). Then, we prove that this contribution of the sing drops discretely which each flip. Hence, flips must terminate

In dimension ≥ 4, there are examples of "smooth" flips, i.e., flips where the flipping locus is contained in the smooth locus of the variety.

Examples of singularities:

Terminal & not smooth: $x^2+y^2+z^2+w^2=0$. (terminal 3-fold).

Canonical & not terminal: $x^2+y^2+z^n=0$.

Klt & not canonical: $\mathbb{C}^2/G$ with $G \leq GL_2(\mathbb{C})$ not in $SL_2(\mathbb{C})$.

dlt & not klt: $(\mathbb{A}^2, H_1+H_2)$

lc & not dlt: Cone over elliptic curve.

rat & not klt: Cone over elliptic / involution

CM + DB & not rat: Cone over elliptic curve.

Quotient & not symplectic: $\mathbb{C}^2/G$ with $G \leq GL_2(\mathbb{C})$ not in $SL_2(\mathbb{C})$.